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Integrability and Pseudointegrability in Billiards Illustrated by the Harmonic Wedge

T. Dagaeff[1]

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We discuss Liouville's theorem for nonsmooth integrable systems of the billiard type and give a scheme of calculation of angle–action variables for the flow. We also deal with the problem of pseudointegrability. We discuss the relationship between the continuous-time (flow) and the discrete-time (map) approaches. We treat all these aspects through a specific billiard—a wedge embedded in a two-dimensional isotropic harmonic potential. Varying the parameters provides two integrable and two pseudointegrable cases. It turns out that the dynamics of one of the latter is indeed integrable in a certain sense. We also address the problem of applying perturbation theory.

KEY WORDS: Billiards; integrability; angle–action variables, pseudointegrability; symplectic maps; Poincaré section.

1. INTRODUCTION

Integrable Hamiltonian systems play a central role in mechanics. They belong to the rare dynamical systems for which we know explicitly the flow. Moreover, thanks to the construction of angle–action variables (AAV), they are the basic systems for application of a perturbation theory and analytical study of some more complicated behaviors.

A way to increase the list of integrable Hamiltonian systems is to consider piecewise integrable systems. They are given by an integrable Hamiltonian flow together with a transformation, say R in phase space which acts at discrete times (reflections on the boundary for billiards or kicks for non-autonomous systems, for instance). We will still say that the system is Hamiltonian if R conserves the energy, but *the dynamics is no longer completely described by a Hamiltonian function.*

[1] Institut de Physique Théorique, École Polytechnique Fédérale de Lausanne, PHB-Ecublens, CH-1015 Lausanne, Switzerland.

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Such systems may exhibit all the features of mechanical systems, from integrability to chaotic behavior. In the generic case, their phase space provides a mixed portrait made of islands of quasiintegrable motion embedded in a sea of stochasticity. This can happen despite integrability of the flow between the times where R acts because it makes the orbit jump from one torus of the integrable motion to another one in a way depending on the state before the transformation. The set of tori reached by one orbit or the order in which it visits them may be very complicated. Hence, to get an integrable system, R must at least conserve constants of motion of the integrable flow. Nevertheless, *the system is still not a smooth Hamiltonian system* and there is no guarantee that all the features of integrable systems will apply to it.

The real point of this paper is to construct angle–action variables (AAV) for a billiard with a nonsmooth boundary, a wedge in a harmonic potential. Many authors who introduced integrable billiards (see refs. 1 and 2, for instance) are interested in proving their integrability, by showing the existence of constants of motion in involution, or in calculating actions in view of the WKB approximation for the quantum billiard. In general, they do not describe in detail the level sets and calculate angle variables, since it is admitted that Liouville's theorem for integrable smooth Hamiltonian systems still applies. This point is not such a trivial one for billiards, since it is not only for pseudointegrable systems that the invariant level sets can be the union of disconnected sheets in phase space. This fact implies certain changes in the formulation of Liouville's theorem, and this renders the calculation of AAV not obvious. The point is to know how the discontinuities due to reflections are taken into account in the AAV formalism.

Moreover, the theorem asserts that the Hamiltonian expressed in the AAV depends only on the action variables. It is interesting to know if this remains true for billiards despite of the nonconnected level sets. Since the perturbation theory of Hamiltonian systems usually deals with the perturbation of the Hamiltonian function itself, it is not evident how to take into account a perturbation of the boundary of billiards. Giving explicitly the Hamiltonian in terms of the actions for our billiard allows us to discuss whether usual perturbation theory can treat modifications of the boundary.

There are two natural ways to deal with piecewise integrable systems. The first is to consider the flow (continuous time). The second is to look at the system only at times when the transformation occurs, by considering a discrete map T including R and the smooth evolution between these times (discrete time). In order to establish certain chaotic properties of a given billiard, it is often simpler to look for the map T rather than for the flow itself. The question of whether the properties of the map (such as Lyapunov exponents, decay of correlation functions,...) reflect correctly the properties of the flow arises naturally (see refs. 3 and 4, for instance). It is then interesting to discuss the relation between the continuous and the discrete approaches even for integrable and pseudointegrable systems, since one can always learn from the simpler situations.

The paper is organized as follows. In Section 2, we address the problem of integrability in piecewise integrable systems of the billiard type in the continuous-time approach. In Section 2.1, we recall Liouville's theorem for integrable smooth Hamiltonian systems. In Section 2.2, we define billiards and explain the main differences with smooth Hamiltonian flows with regard to the nature of level sets. We investigate what difficulties can arise when applying Liouville's theorem to such systems. We illustrate these considerations by constructing angle–action variables for the rectangular billiard and by underlining the existence of what Richens and Berry called "pseudointegrable" systems.[5] We describe what changes must be made in the theorem and in the construction of an AAV scheme for the smooth case to make them still valid for piecewise smooth Hamiltonian systems. In Section 2.3, we introduce a billiard with potential, the "harmonic wedge"—a wedge embedded in a two-dimensional isotropic harmonic potential, and apply our scheme of construction of AAV to two integrable cases of very different kinds. We make some comments on the possibility of applying KAM theory to the harmonic wedge by use of the construction of AAV for both cases. In Section 2.4, we present a pseudointegrable case of this billiard.

In Section 3, we address the problem of integrability in the discrete-time approach. In Section 3.1, we recall Liouville's theorem for symplectic diffeomorphisms. In Section 3.2 we point out what problems could arise when the map we consider is the bouncing map of a billiard with non-smooth boundary. In Section 3.3, we ask what the relationship is between the AAV obtained for the continuous time and the AAV obtained for the discrete time. We construct AAV for the bouncing map of the harmonic wedge for the two integrable cases described in Section 2.3. This allows us to discuss the case where the bouncing map is not a diffeomorphism and the case where it is a well-defined Poincaré section. In Section 3.4 we investigate what distinguishes the pseudointegrable case from the integrable one in this approach.

2. THE CONTINUOUS-TIME APPROACH

2.1. Liouville's Theorem for Integrable Smooth Hamiltonian Systems

We recall Liouville's theorem for integrable smooth Hamiltonian systems. Consider a system of n degrees of freedom with the Hamiltonian function $H^0(\mathbf{p}, \mathbf{q})$ which induces the flow Φ_t^0 on the phase space M^0. Suppose that there are n functions $F_1,..., F_n$ on M^0 which are integrals of motion in involution:

$$\{F_i, H^0\} = 0 \quad \forall i, \quad \text{which implies } F_i(\Phi_t^0 \mathbf{x}) = F_i(\mathbf{x}) \quad \forall \mathbf{x} \in M^0$$

$$\{F_i, F_j\} = 0 \quad \forall i, j$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. We will call such functions "constants of motion." Let $M_{\mathbf{k}}^0$ (sometimes denoted by $M^0_{(F_1 = k_1,..., F_n = k_n)}$ in what follows) be a level set of these functions:

$$M_{\mathbf{k}}^0 = \{\mathbf{x} \in M^0 | F_i(\mathbf{x}) = k_i, i = 1,..., n\}$$

and assume that the n 1-forms dF_i are independent at each point of $M_{\mathbf{k}}^0$. The theorem asserts:[6]

(i) $M_{\mathbf{k}}^0$ is smooth and invariant under Φ_t^0.

(ii) If $M_{\mathbf{k}}^0$ is compact and connected, then it is diffeomorphic to the n-dimensional torus $\mathbb{T}^n = \{(\varphi_1,..., \varphi_n) \bmod 2\pi\}$.

(iii) Canonical equations of motion may be integrated by quadratures to give $(d/dt)\, \boldsymbol{\varphi} = \boldsymbol{\omega}(\mathbf{k})$.

In view of Liouville's theorem, we will say that such a system is integrable (or briefly in what follows, that H^0 or Φ_t^0 is integrable). The angles φ_i and their conjugate momenta I_i are called "angle–action variables" (AAV). The canonical change of variables from $(\mathbf{p}, \mathbf{q})$ to $(\mathbf{I}, \boldsymbol{\varphi})$ can be done explicitly with the help of the following procedure:

Scheme 1

1. Find the level sets $M_{\mathbf{k}}^0$ associated to the n constants of motion.
2. Find n one-dimensional independent cycles Γ_i on the torus $M_{\mathbf{k}}^0$.
3. Calculate the actions given by

$$I_i(\mathbf{k}) = \frac{1}{2\pi} \oint_{\Gamma_i} \mathbf{p}(\mathbf{k}, \mathbf{q})\, d\mathbf{q}, \qquad i = 1,..., n$$

4. Calculate the generating function

$$S(\mathbf{I}, \mathbf{q}) = \int_{\mathbf{q}_0}^{\mathbf{p}} \mathbf{p}(\mathbf{k}(\mathbf{I}), \mathbf{q}')\, d\mathbf{q}'$$

5. Calculate the angles from $\boldsymbol{\varphi} = (\partial/\partial \mathbf{I})\, S$.

There are two main steps in the proof of this theorem. First, one constructs n vector fields $\mathbf{V}_i$ by calculating the symplectic gradient of the constants of motion:

$$\mathbf{V}_i = J\nabla F_i \equiv \left(\frac{\partial}{\partial \mathbf{p}} F_i, \; -\frac{\partial}{\partial \mathbf{q}} F_i\right)$$

Using involution, one can prove that these vector fields are linearly independent, tangent to $M^0_{\mathbf{k}}$, and commuting. The second step consists in proving a theorem which asserts that an n-dimensional smooth manifold on which are defined n vector fields having these properties is diffeomorphic to an n-dimensional torus. Points 2–5 of Scheme 1 are applicable to any torus in a symplectic space.

We will see now how to adapt this theorem, and in particular the above scheme for constructing AAV, to piecewise integrable Hamiltonian systems.

2.2. Problem of Billiards

Billiards are dynamical systems corresponding to the motion of a pointlike particle in a domain $Q \subset Q^0$. The point reflects from the boundary ∂Q according to the law of elastic reflection and its motion between collisions is defined by the geodesic flow Φ^0_t associated to a Hamiltonian function H^0 with Q^0 as configuration space. Let M be the phase space of the billiard, and ∂M the set of points $(\mathbf{p}, \mathbf{q}) \in M$ such that $\mathbf{q} \in \partial Q$ and $\mathbf{p}$ is oriented toward the exterior of Q. We assume that Q^0 is two-dimensional, not necessarily compact, but that the manifolds of constant energy of the billiard are compact.

A billiard is a piecewise integrable Hamiltonian system if the Hamiltonian H^0 on phase space M^0, i.e., the system without collisions, is integrable: there are two constants of motion between collisions. In general, it is not possible to find an explicit expression for the flow Φ_t of the billiard, since we have to iterate the following discrete algorithm in order to obtain a solution:

(i) Starting from $\mathbf{x}_n \in \partial M$ at time t_n, apply the reflection: $\mathbf{x}_n \mapsto R\mathbf{x}_n$.

(ii) Compute t_{n+1}, the smallest time for which the trajectory issued from $R\mathbf{x}_n$ reaches the boundary.

(iii) Get the evolution from t_n to t_{n+1} by the geodesic flow starting from $R\mathbf{x}_n$: if $t_n < t \leqslant t_{n+1}$, then $\mathbf{x}(t) = \Phi^0_{t-t_n}(R\mathbf{x}_n)$ and $\mathbf{x}_{n+1} = \mathbf{x}(t_{n+1})$.

We will denote such a system by $H^0 \circ$ *reflections.*

When and in what sense will such a system be integrable? In other words, what could go wrong when we try to apply Scheme 1 to billiards?

At a reflection, the orbit of the billiard generally jumps from one invariant two dimensional manifold (a torus if it is compact) of the integrable smooth geodesic flow to another. The manifold which is reached and on which the motion takes place before the next encounter with the boundary is determined by the state of the system before the reflection. By this process, the orbit may explore the whole manifold of constant energy in M. To obtain an integrable system, we have obviously to find constants of motion of the geodesic flow which remain constant at the collisions. But it is still possible that the orbit of the billiard reaches several (a discrete number) manifolds each one invariant under the geodesic flow. Hence, the constants of motion that we have to choose for the billiard are not necessarily the ones that lead to connected level sets in the case without collisions. The level sets $M^0_{\mathbf{k}}$ in M^0, for the same constants of motion, may be the union of several smooth surfaces each one invariant under Φ^0_t. In this case, the level set $M_{\mathbf{k}} = \{\mathbf{x} \in M \mid F_i(\mathbf{x}) = k_i,\ i = 1,..., n\}$ for the billiard is the union of disconnected pieces of $M^0_{\mathbf{k}}$. From the preceding, and since their configuration spaces are different, $M_{\mathbf{k}}$ is not identical to $M^0_{\mathbf{k}}$. We lose the smoothness of the level sets and the first assertions of Liouville's

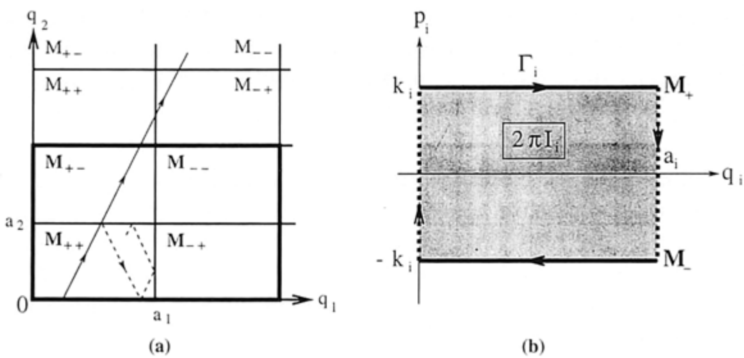

Fig. 1. (a) Unfolding of a trajectory of the rectangular billiard. We perform successive symmetries of the original rectangle M_{++} about its sides so that the original trajectory (dashed line) is represented by a straight line. The four possible values of the monumentum along the original trajectory correspond to the passage of the unfolded one through the copies of the original rectangle denoted M_{++}, M_{+-}, M_{-+}, and M_{--}. The bold rectangle represents the torus $M^c_{(k_1, k_2)}$. (b) One independent cycle Γ_i. The solid lines are associated with the free motion between collisions and the dashed lines correspond to the jump due to the reflection. The area enclosed by this cycle gives the action I_i.

theorem become wrong. To manage this difficulty we will paste smoothly the disconnected parts of $M_{\mathbf{k}}$ into a new surface $M_{\mathbf{k}}^c$, and check if the remainder of the theorem still applies to this invariant manifold.

Calculation of the AAV for the well-known rectangular billiard illustrates the preceding remarks. The Hamiltonian function $H^0 = \frac{1}{2}(p_1^2 + p_2^2)$ describes a free particle on a plane and Q is a rectangle with sides of length a_1 and a_2.

In this particular case where the billiard tiles the entire plane, the calculation of AAV must reproduce the results of the geometrical construction which consists in unfolding the trajectory (Fig. 1a) and takes advantage of the fact that the billiard tiles the entire plane (see ref. 7 for instance).

We can separate each direction $i = 1, 2$. Constants of motion are the functions $|p_i|$, $i = 1, 2$.

1. The level sets are given by

$$M_{\mathbf{k}} \equiv M_{(|p_1| = k_1,\, |p_2| = k_2)} = M_{++} \cup M_{+-} \cup M_{-+} \cup M_{--}$$

where

$$M_{\mu\nu} = \{(\mathbf{p}, \mathbf{q}) \in M \mid p_1 = \mu k_1,\ p_2 = \nu k_2,\ q_1 \in [0, a_1[,\ q_2 \in [0, a_2]\}$$

Since $M^0_{(k_1, k_2)}$ is the union of four infinite planes each one invariant under Φ^0_t, the motion of the free particle (without boundary) would take place on one of these planes only, but the particle in the billiard at the collision jumps from one of the planes to another. This illustrates the above remarks on the nature of the level sets.

2. Consequently, an independent cycle Γ_i crosses the discontinuity corresponding to one reflection: it is an orbit in one dimension (Fig. 1b).

3. We calculate the actions as usual from the areas enclosed by the cycles in the $(\mathbf{p}, \mathbf{q})$ space. We get $I_i = k_i a_i / \pi$.

4, 5. Since we have to consider disconnected areas of phase space, we expect to calculate different expressions of the generating function for each smooth component of $M_{\mathbf{k}}$. In one dimension we have

$$\text{if} \quad p_i = k_i, \qquad S(I_i, q_i) = \pi I_i \frac{q_i}{a_i}, \qquad \varphi_i = \pi \frac{q_i}{a_i}$$

$$\text{if} \quad p_i = -k_i, \qquad S(I_i, q_i) = \pi I_i \left(2 - \frac{q_i}{a_i}\right), \qquad \varphi_i = 2\pi - \pi \frac{q_i}{a_i}$$

We get a single expression for the frequencies:

$$\frac{d}{dt}\varphi_i = \omega(I_i) = \pi^2 \frac{I_i}{a_i^2}$$

As expected, these results correspond to the definition of the torus $M_{\mathbf{k}}^c$ obtained in the geometrical approach. But there are two advantages in using this construction. The first is that we have just to follow a procedure that can be applied to any integrable billiard. The second is that we obtain a Hamiltonian function depending on the actions only. Indeed, by construction of $M_{\mathbf{k}}^c$ the variables $(\mathbf{I}, \boldsymbol{\varphi})$ are not changed at a reflection, so that

$$H^0(\mathbf{p}, \mathbf{q}) \circ \textit{reflections} \Leftrightarrow H(\mathbf{I}) = \frac{\pi^2}{2}\left(\frac{I_1^2}{a_1^2} + \frac{I_2^2}{a_2^2}\right)$$

The two above expressions of (φ_1, φ_2) together with the expression of (I_1, I_2) (after replacing k_i by $|p_i|$) define an isomorphism from $(\mathbf{p}, \mathbf{q})$ to $(\mathbf{I}, \boldsymbol{\varphi})$. This implies a first change to be made in Liouville's theorem: since we pasted the disconnected parts of $M_{\mathbf{k}}$, this level set is only isomorphic to a torus, and we have lost differentiability of the change of variables.

It is important to note why the construction by quadrature works for the rectangular billiard. The crucial step is the characterization of $M_{\mathbf{k}}^c$. If it is diffeomorphic to a torus, the second step of the proof of Liouville's theorem, applied to it instead of $M_{\mathbf{k}}^0$, is still valid and we can use without changes points 2–5 of Scheme 1.

If it is not, this can be one obstacle to the validity of Liouville's theorem. This difficulty was underlined by Richens and Berry.[5] They introduced a billiard ("square torus billiard") which has two constants of motion but which is not integrable. The trouble comes from the fact that in this case the surface $M_{\mathbf{k}}^c$ is diffeomorphic to a five-handled sphere. Indeed, the pasting of the components of $M_{\mathbf{k}}$ may introduce singularities of nonzero index in the vector fields associated with the constants of motion, so that the genus of the resulting surface is different from 1. The presence of these singularities prevents us from applying Liouville's theorem, which assumes that the vector fields $J\nabla F_i$ are not singular: there are no global angle–action variables. Following Richens and Berry, we will define a system of two degrees of freedom having two constants of motion in involution and for which the phase space is foliated in two-dimensional invariant manifolds isomorphic to multiply handled spheres as a "pseudointegrable" system. Note that the paper of Richens and Berry does not deal with integrable billiards: it does not discuss the consequences of pasting the level sets in integrable cases for the calculation of angle–action variables and for the expression of the Hamiltonian in terms of the actions only.

To sum up what we learned from the above example and from the counterexample, we have to change Scheme 1 slightly in order to establish an AAV calculation scheme applicable to other billiards.

Scheme 2

1. Find the level sets $M_{\mathbf{k}}$ associated to the n constants of motion.

1′. Characterize $M_{\mathbf{k}}$, which may be piecewise smooth, being the union of parts of tori of the motion without collisions. Paste the disconnected parts of $M_{\mathbf{k}}$ to obtain $M_{\mathbf{k}}^c$ in such a way that the symplectic gradients of the constants of motion will form continuous vector fields on $M_{\mathbf{k}}^c$. Verify if this surface is diffeomorphic to a torus. This can be checked by calculating the index of eventual singular points of the vector fields or geometrically by correct identification of edges and vertices of the sheets in phase space.

2. Find n one-dimensional independent cycles Γ_i on the torus $M_{\mathbf{k}}^c$. These cycles may be unions of smooth curves each one lying on disconnected parts of $M_{\mathbf{k}}$.

3. Calculate the actions given by the areas

$$2\pi I_i = \oint_{\Gamma_i} \mathbf{p}(\mathbf{k}, \mathbf{q})\, d\mathbf{q}$$

which is a sum over the continuous pieces of Γ_i.

4. Calculate the generating function, which has a different expression on each continuous piece of the level set.

5. Calculate the angles $\boldsymbol{\varphi}$ by differentiation of the different expressions of the generating function.

We then obtain the frequencies $\boldsymbol{\omega}(\mathbf{I})$ and the Hamiltonian function $H(\mathbf{I})$ depending on the actions only. Since by construction of $M_{\mathbf{k}}^c$ the variables $(\mathbf{I}, \boldsymbol{\varphi})$ are not changed at a reflection, the resulting Hamiltonian $H(\mathbf{I})$ contains the whole dynamics of the system: the geodesic flow *and* the reflections.

Following this scheme, we will now construct AAV for the two integrable cases of a billiard with potential.

2.3. AAV for the Flow of the Harmonic Wedge

2.3.1. Description. The "harmonic wedge" is characterized by (Fig. 2):

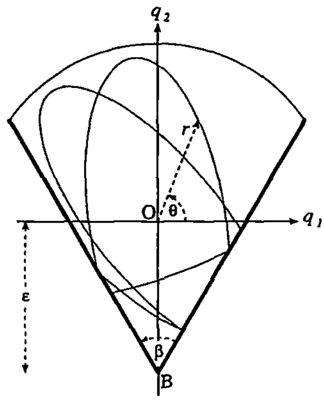

Fig. 2. The harmonic wedge. The boundary is made of two straight lines intersecting at B with an angle β. There is an isotropic harmonic potential centered at O at a distance ε from B on the bisector of the angle. Trajectories are made of pieces of ellipses. The arc of a circle is the location of the maximum of potential energy: it bounds the motion in the q_2 direction. Definition of polar coordinates (r, θ) is indicated.

(i) The Hamiltonian including an isotropic harmonic potential:

$$H^0 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + r^2\right)$$

Equipotentials are circles centered on the origin O. This Hamiltonian is invariant under rotations. Trajectories between collisions are arcs of ellipses.

(ii) The boundary, two straight lines intersecting with an angle β at B.

(iii) The position of O, at a distance ε from B on the bisector of the angle. We will take $\varepsilon > 0$ if O lies inside the accessible domain Q (as in Fig. 2) and $\varepsilon < 0$ if O lies outside Q.

The motion is bounded by the boundary and by the potential.

This billiard is isomorphic to the system of three particles on a line interacting with harmonic potentials. This can be seen by performing the standard calculation for the linear chain in solid-state physics. The main steps are the following. First, introduce a canonical transformation which makes the kinetic energy isotropic when the masses of the particles are not equal. Second, write the Hamiltonian in the normal modes. The corresponding billiard system is then obtained by freezing the center of gravity.

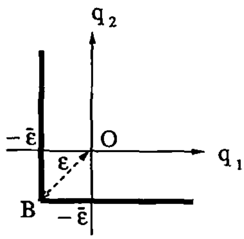

Fig. 3. Coordinate system (q_1, q_2) chosen when $\beta = \pi/2$.

The sides of the wedge represent the collisions between particles. The corner represents triple collisions. The angle β depends on the masses of the three particles and the distance ε depends on the distances between the particles at the equilibrium position.

Varying the total energy $E = H$ is equivalent to varying ε. So we fix $E = 1$. Note that we will consider only the invariant subset of phase space corresponding to "proper" orbits of the billiard, i.e., orbits having collisions with the boundary (an orbit having a collision has infinitely many collisions). The other, nonproper, orbits belong to an invariant domain of the phase space in which the system is simply one of the two noncoupled oscillators described by H^0.

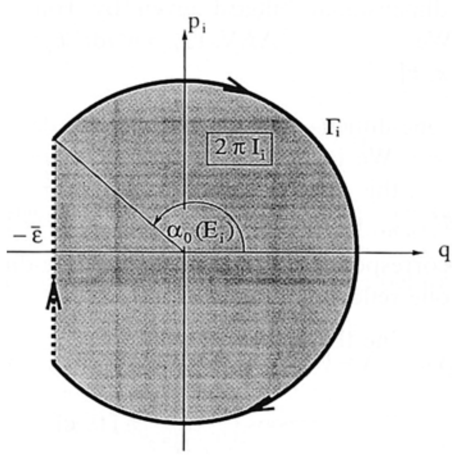

Fig. 4. One independent cycle Γ_i of the harmonic wedge with $\beta = \pi/2$. The arc of a circle of radius $(2E_i)^{1/2}$ is associated with the harmonic motion between collisions and the dashed line corresponds to the jump at the reflection. The area enclosed by this cycle gives the action I_i. Definition of the angle $\alpha_0(E_i)$ is indicated.

We will now treat the following two integrable cases according to Scheme 2.

(i) $\beta = \pi/2$ with any $\varepsilon \neq 0$.

(ii) $\varepsilon = 0$ with any β.

2.3.2. Integrable Case $\beta = \pi/2$. It is convenient to work in the coordinate system of Fig. 3, where q_1 is parallel to one of the sides (which we will call side 1) and q_2 to the other side (side 2). We define the parameter $\bar{\varepsilon} \equiv \varepsilon/\sqrt{2}$. The energies $h_i^0 = \frac{1}{2}(p_i^2 + q_i^2)$, $i = 1, 2$, of both modes are constant. The system remains separable, which makes the calculation very simple. The level sets are given by $h_1^0 = E_1$ and $h_2^0 = E_2$. If $\varepsilon > 0$, we have to distinguish between two cases: (a) $2E_i \leqslant \bar{\varepsilon}^2$ for $i = 1$ or 2 (there are no collisions with side j, $j \neq i$), (b) $2E_i > \varepsilon^2$ for both $i = 1$ and 2. The phase space is the union of two invariant subset $M^{(a)}$ and $M^{(b)}$ corresponding to these two cases. Note that there are also trivial cases, if ε is sufficiently big, for which all orbits are nonproper: $M_{(E_1, E_2)} = M^0_{(E_1, E_2)}$ and we can use the AAV of the system without collisions. The next calculations are done for case (b).

1. We have that

$$M_{(E_1, E_2)} \equiv M_{(h_1^0 = E_1,\, h_2^0 = E_2)}$$

is a piece of the torus $M^0_{(E_1, E_2)}$. We can reduce the calculation to the study of the one-dimensional billiard given by Hamiltonian h_i^0 and a wall at $q_1 = -\bar{\varepsilon}$. We introduce AAV (J_i, α_i) for h_i^0: $J_i = -E_i$ and $\alpha_i = \arctan(p_i/q_i) \in [-\pi, \pi[$.

1′. For each one-dimensional system, we can define the vector field $V_i = J\nabla h_i^0 = (p_i, -q_i)$. We identify the point $(q_i, p_i) = (-\bar{\varepsilon}, -\sqrt{2E_i})$ with the point $(-\bar{\varepsilon}, \sqrt{2E_i})$: this applies cycle $\Gamma_i \equiv M_{E_i}$ of Fig. 4 on a circle. The resulting surface $M^c_{(E_1, E_2)}$ is diffeomorphic to $S^1 \otimes S^1 \equiv \mathbb{T}^2$.

2. Cycle: Γ_i corresponds to an orbit of the one-dimensional system, and then includes one reflection (Fig. 4).

3. Action: we define the angle

$$\alpha_0(E_i) = \arccos\left(\frac{-\bar{\varepsilon}}{\sqrt{2E_i}}\right) \in [0, \pi[$$

then

$$I_i = \frac{E_i}{2\pi}\{2\alpha_0(E_i) - \sin[2\alpha_0(E_i)]\}$$

4. Generating function:

$$\text{if} \quad p_i > 0, \qquad S(I_i, q_i) = \int_{-\bar{\varepsilon}}^{q_i} p(E_i, q)\, dq$$

$$= -2E_i \int_{-\alpha_0(E_i)}^{\alpha_i} \sin^2(\alpha)\, d\alpha$$

$$\text{if} \quad p_i < 0, \qquad S(I_i, q_i) = \pi I_i + \int_0^{q_i} p(E_i, q)\, dq$$

$$= \pi I_i - 2E_i \int_0^{\alpha_i} \sin^2(\alpha)\, d\alpha$$

Then

$$S(I_i, q_i) = \pi I_i + \frac{E_i}{2} [\sin(2\alpha_i) - 2\alpha_i]$$

with

$$\alpha_i = \alpha(E_i, q_i) = \pm \arccos\left(\frac{q_i}{\sqrt{2E_i}}\right) \qquad \text{and} \qquad E_i = E(I_i)$$

5. Angle:

$$\phi_i = \frac{\partial S}{\partial I_i} = -\frac{\pi\alpha_i}{\alpha_0(E_i)} \in [-\pi, \pi[$$

We get

$$\frac{d}{dt}\phi_i = \omega(I_i) = \frac{\pi}{\alpha_0(E(I_i))}$$

and

$$H^0(\mathbf{p}, \mathbf{q}) \circ \textit{reflections} \Leftrightarrow H(\mathbf{I}) = E(I_1) + E(I_2)$$

Case (a) leads simply to I_j and φ_j defined as before, $I_i = -E_i$, $\varphi_i = \alpha_i$, and

$$H^0(\mathbf{p}, \mathbf{q}) \circ \textit{reflections} \Leftrightarrow H(\mathbf{I}) = I_i + E(I_j)$$

Due to the distinction between case (a) and case (b), the change of variables leading to AAV is not the same in the whole phase space. Geometrically the construction of AAV corresponds to rescaling the angular variables α_i so that the piece of the torus $M^0_{(E_1, E_2)}$ on which the motion with collisions takes place is mapped onto a torus (Fig. 5). Indeed,

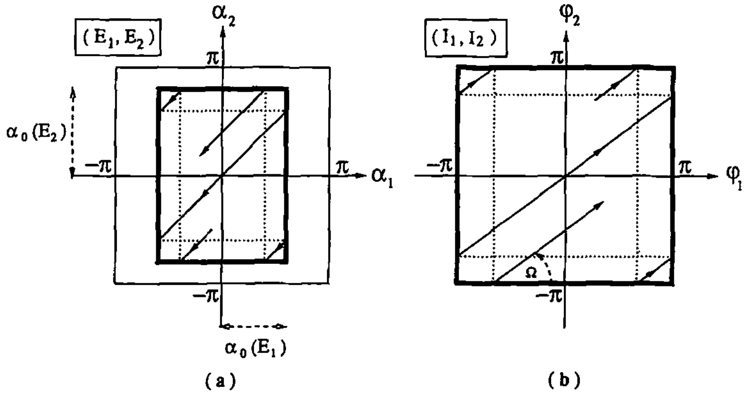

Fig. 5. (a) The torus diffeomorphic to $M^0_{(E_1,E_2)}$ of the two noncoupled harmonic oscillators defined by (E_1, E_2) and parametrized by (α_1, α_2). The bold box represents the boundary of the harmonic wedge with $\beta = \pi/2$. The arrowed line of slope 1 is an orbit. The dotted lines represent the reflections. (b) The torus diffeomorphic to the level set $M_{(E_1,E_2)}$ of the harmonic wedge with $\beta = \pi/2$ defined by (I_1, I_2) and parametrized by (φ_1, φ_2). It is obtained by rescaling the bold box of (a). The orbits become straight lines of slope tan $(\Omega) = \alpha_0(E(I_1))/\alpha_0(E(I_2))$.

the side i is given by $q_i \in [-\bar{\varepsilon}, \sqrt{2E_i}]$ and $q_j = -\bar{\varepsilon}$, so that, in variables (α_1, α_2) at fixed values of (E_1, E_2), $\alpha_i \in [-\alpha_0(E_i), \alpha_0(E_i)]$ and $\alpha_j = \pm\alpha_0(E_j)$, where $j \neq i$ (minus sign corresponding to ∂M, i.e., to our convention of considering the outside-oriented momentum on the boundary).

The motion takes place in the interior of the rectangle limited by the sides, and the reflections identify opposite faces of this rectangle. This gives us another way to see that $M_{(E_1,E_2)}$ is diffeomorphic to a torus.

This case is simpler than that of the rectangle, since we do not have to paste disconnected areas of M. Note that here the billiard also tiles the plane.

2.3.3. Integrable Case $\epsilon = 0$. In this case the center of the potential O is at the vertex B. The absolute value of the angular momentum p_θ is constant. We take H^0 and $|p_\theta|$ as the two constants of motion. Note that we could choose the constant of motion to be p_θ^2. We prefer the absolute value, because choosing p_θ^2 leads to a level set $M^0_{(H^0=E,\,p_\theta^2=L^2)}$ which is, even in the case without collisions, the union of two disconnected surfaces each one invariant under the flow and diffeomorphic to a torus. Hence the natural parameter of the tori (or, here, parts of tori) is the value of the angular momentum.

1. We have

$$M_{(E,|L|)} \equiv M_{(H^0=E,\,|p_\theta|=|L|)} = M_{(H^0=E,\,p_\theta=|L|)} \cup M_{(H^0=E,\,p_\theta=-|L|)}$$

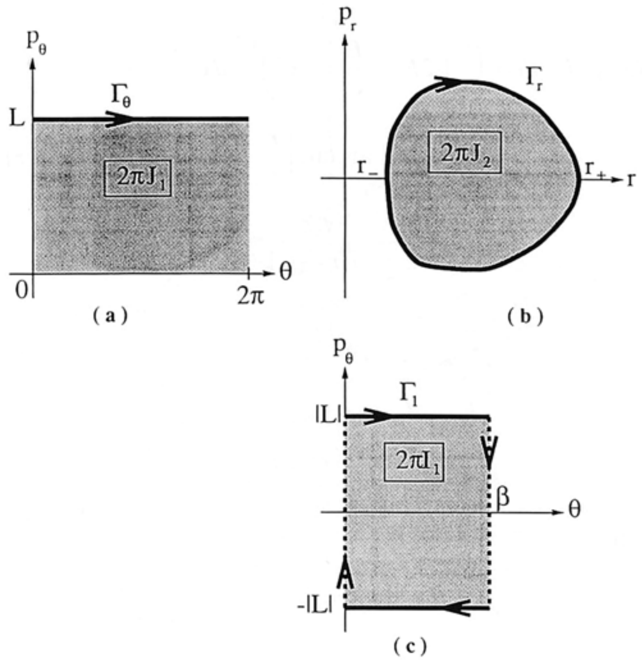

Fig. 6. (a) Cycle Γ_θ on $M^0_{(E,L)}$. (b) Cycle Γ_r on $M^0_{(E,L)}$. (c) Cycle Γ_1 on $M_{(E,|L|)}$ of the wedge.

$M^0_{(E,|L|)}$ is the union of two disconnected tori each one invariant under Φ^0_t and $M_{(E,|L|)}$ is the union of two pieces of each torus constituting $M^0_{(E,|L|)}$. It is then convenient to introduce AAV $(\mathbf{J}, \boldsymbol{\alpha})$ of H^0, considering H^0 and p_θ as constants of motion and denoting by $M^0_{(E,L)} \equiv M^0_{(H^0=E,\, p_\theta=L)}$ the (connected) level set diffeomorphic to a torus:

- Cycles: Γ_θ and Γ_r (Figs. 6a and 6b). Γ_r is an orbit of

$$h^0_r(p_r, r) = \frac{1}{2}\left(p_r^2 + \frac{L^2}{r^2} + r^2\right)$$

- Actions:

$$J_1 = L \qquad \text{and} \qquad J_2 = \frac{E - |L|}{2}$$

- Generating function:

$$S(\mathbf{J}, r, \theta) = J_1\theta + S_r(J_1, J_2, r)$$

where

$$S_r(J_1, J_2, r) = \int_{r_-}^{r} \left(2E - \frac{|L|}{x^2} - x^2\right)^{1/2} dx$$

$$= \pi J_2 + |J_1| \arctan\left[\frac{r_-}{r_+} t(r)\right] - E \arctan[t(r)]$$

$$+ \frac{r_+^2 - r_-^2}{2} \frac{t(r)}{t(r)^2 + 1}$$

and

$$r_\pm^2 = E \pm (E^2 - L^2)^{1/2}$$

$$E = 2J_2 + |J_1|$$

$$t(r) = \frac{[-(r^2 - r_-^2)(r^2 - r_+^2)]^{1/2}}{r^2 - r_-^2}$$

- Angles:

$$\alpha_1 = \frac{\partial S}{\partial J_1} = \theta + \operatorname{sign}(p_\theta) \arctan\left(-\frac{rp_r}{r^2 + |L|}\right)$$

$$\alpha_2 = \frac{\partial S}{\partial J_2} = \arctan\left(-\frac{rp_r}{r^2 - E}\right)$$

- Frequencies:

$$\frac{d}{dt}\alpha_1 = \omega(J_1) = \operatorname{sign}(J_1) \qquad \text{and} \qquad \frac{d}{dt}\alpha_2 = \omega(J_2) = 2$$

1′. As we can see in Fig. 7, $M_{(E, |L|)}$ is the union of two surfaces, each one a part of a tube diffeomorphic to a truncated cylinder. Indeed, the vector field associated to H^0 is, in coordinates $(r, \theta, p_r, p_\theta)$, $\mathbf{V}_1 = J\nabla H^0 = (p_r, p_\theta/r^2, p_\theta/r^3 - r, 0)$ and the one associated to $|p_\theta|$ is $\mathbf{V}_2 = J\nabla|p_\theta| = (0, \operatorname{sign}(p_\theta), 0, 0)$. We can clearly paste the two parts of $M_{(E, |L|)}$ without introducing singularities in these vector fields to obtain $M^c_{(E, |L|)}$ which is diffeomorphic to a torus.

2. Cycles: Γ_1 includes one reflection and Γ_2 is identical to Γ_r (Fig. 6c).

3. Actions:

$$I_1 = \frac{\beta}{\pi}|L| = \frac{\beta}{\pi}|J_1| \qquad \text{and} \qquad I_2 = \frac{E - |L|}{2} = J_2$$

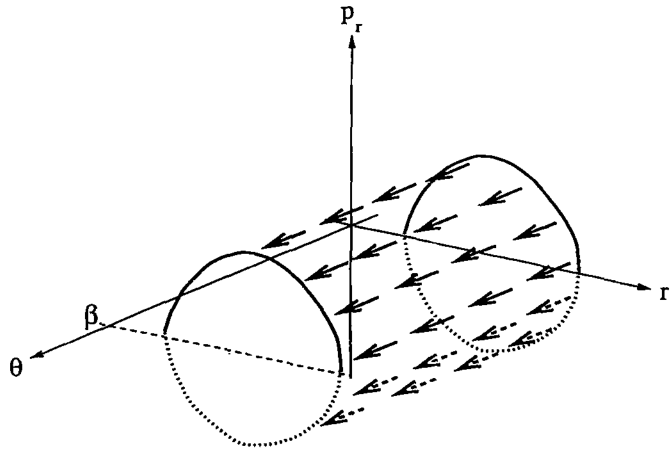

Fig. 7. The piece of $M_{(E,|L|)}$ with $p_\theta > 0$: the projection in plane (p_r, r) is an orbit of $h_r^0(p_r, r) = \frac{1}{2}(p_r^2 + L^2/r^2 + r^2)$ (see the cycle Γ_r of Fig. 6b). The arrows represent the vector field $J\nabla|L|$. If $p_\theta < 0$, the arrows would be oriented in the opposite direction (of decreasing θ) and we would get a picture of the second piece of $M_{(E,|L|)}$. These two pieces can be smoothly connected to form a torus.

4. Generating function: we have to consider two expressions of S:

on $M_{(H^0=E,p_\theta=|L|)}$, $$S(I_1, I_2, r, \theta) = \frac{\pi}{\beta} I_1 \theta + S_r\left(\frac{\pi}{\beta} I_1, I_2, r\right)$$

on $M_{(H^0=E,p_\theta=-|L|)}$, $$S(I_1, I_2, r, \theta) = \frac{\pi}{\beta} I_1 (2\beta - \theta) + S_r\left(\frac{\pi}{\beta} I_1, I_2, r\right)$$

5. Angles:

on $M_{(H^0=E,p_\theta=|L|)}$, $\varphi_1 = \frac{\pi}{\beta}\alpha_1 \in [0, \pi[$, $\varphi_2 = \alpha_2$

on $M_{(H^0=E,p_\theta=-|L|)}$, $\varphi_1 = -\frac{\pi}{\beta}\alpha_1 \in [-\pi, 0[$, $\varphi_2 = \alpha_2$

Then

$$\varphi_1 = \operatorname{sign}(J_1)\frac{\pi}{\beta}\alpha_1 \in [-\pi, \pi[, \qquad \varphi_2 = \alpha_2$$

We get

$$\frac{d}{dt}\varphi_1 = \omega(I_1) = \frac{\pi}{\beta}, \qquad \frac{d}{dt}\varphi_2 = \omega(I_2) = 2$$

and

$$H^0(\mathbf{p}, \mathbf{q}) \circ \mathit{reflections} \;\Leftrightarrow H(\mathbf{I}) = 2I_2 + \frac{\pi}{\beta} I_1$$

We had to paste parts of tori of H^0, as for the rectangular billiard. The particular interest of this example is that the billiard does not necessarily tile the plane: the surface $M^c_{(E, |L|)}$ is diffeomorphic to a torus for any β. The construction of AAV is more general than the geometrical one, which consists in unfolding the trajectory. We point out that the vector fields can be nonsingular, even if the boundary has a nonrational angle. Here the preimages of the singularity of the boundary are not dense in phase space (they lie on a manifold of codimension 1, which is singular for the change of variables, even in the case without collisions). Hence the orbits are not affected by the singularity, whereas in the case of the square torus billiard these preimages are dense and almost all orbits are separated by the discontinuity.

2.3.4. Perturbation of the Boundary. Now the question arises of whether the single Hamiltonian formulation including reflections is preserved when we perturb the integrable situations. The answer differs for the two cases.

In the case $\beta = \pi/2$, we may perturb the system by changing the angle β. Unfortunately, when we do this, the reflections expressed in the AAV are no longer the identity, so that we are forced to handle explicitly the collisions and we lose the structure introduced above: the system described by $H^{(\beta=\pi/2)}(\mathbf{I})$ becomes $H^{(\beta=\pi/2+\delta)}(\mathbf{I}, \varphi) \circ \mathit{reflections}$.

In the case $\varepsilon = 0$ we may perturb the system by changing the value of ε. Since $L = 0$ is a singular value for the change of variables, we cannot treat orbits having vanishing angular momentum at a certain time through the AAV introduced for the integrable case. This corresponds to orbits passing through B or being tangent to the boundary. But we can use the same variables for the other orbits which define an invariant set of nonzero measure in phase space, for most values of ε and β, made of those orbits going directly from one side to the other. So we get a perturbed Hamiltonian

$$H^{(\varepsilon)}(\mathbf{I}, \boldsymbol{\varphi}) = 2I_2 + \frac{\beta}{\pi} I_1 + \frac{\varepsilon}{2} r(\mathbf{I}, \boldsymbol{\varphi}) \sin(\theta(\mathbf{I}, \boldsymbol{\varphi}))$$

without an additional reflection rule, which describes this particular type of orbits. The study of this Hamiltonian should allow us to get information on the corresponding restriction of the phase space. Unfortunately, the very complicated form of the potential in AAV did not allow us to pursue this direction. Note that in both cases we may also perturb the system by making the potential anisotropic.

2.4. Pseudointegrable Case $\beta = 3\pi/2$

The situation and definition of (q_1, q_2) are shown in Fig. 8. We have the same constants of motion as in the case $\beta = \pi/2$.

The shorter way to check that this system is pseudointegrable is to examine the motion of the billiard point using coordinates (α_1, α_2) at fixed values of (E_1, E_2), i.e., on a torus $M^0_{(E_1, E_2)}$ of the motion without collisions.

The side i (parallel to axis q_i) is given by $q_i \in [\bar{\varepsilon}, \sqrt{2E_i}]$ and $q_j = \bar{\varepsilon}$, so that, in variables (α_1, α_2) at fixed values of (E_1, E_2), $\alpha_i \in [-\bar{\alpha}_i, \bar{\alpha}_i]$ and $\alpha_j = \pm\bar{\alpha}_j$, where $j \neq i$ and $\bar{\alpha}_i = \arccos[\bar{\varepsilon}/(2E_i)^{1/2}] \in [0, \pi]$, $i = 1, 2$ (plus sign for α_j corresponds to ∂M, i.e., to our convention of considering the outside-oriented momentum on the boundary).

The motion takes place at the exterior of the rectangle limited by the sides, and the reflections identify the opposite faces of this rectangle (see Fig. 9). It is easy to check that the resulting surface $M_{(E_1, E_2)}$ with these identifications is isomorphic to a two-handled sphere, so that the system is pseudointegrable in the sense of Richens and Berry.[5]

Although the energies of the two modes are conserved, the system is not separable, due to collisions. Indeed, when we follow the evolution of coordinates (p_1, q_1), we have to invert p_1 as one collision with slide 2 occurs, i.e., when $q_1 = \bar{\varepsilon}$ and $p_1 > 0$ if, and only if, at the same time $q_2 > \bar{\varepsilon}$. When $\beta = \pi/2$ we did not need to follow the evolution of q_2 to know when to invert p_1: this is the difference between the two cases.

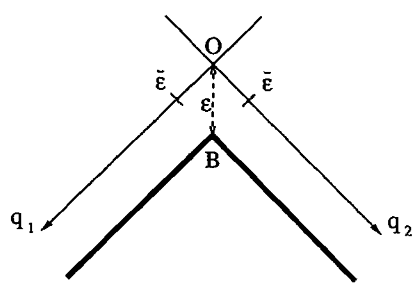

Fig. 8. Coordinate system (q_1, q_2) chosen when $\beta = 3\pi/2$. Here O is inside the billiard ($\varepsilon > 0$).

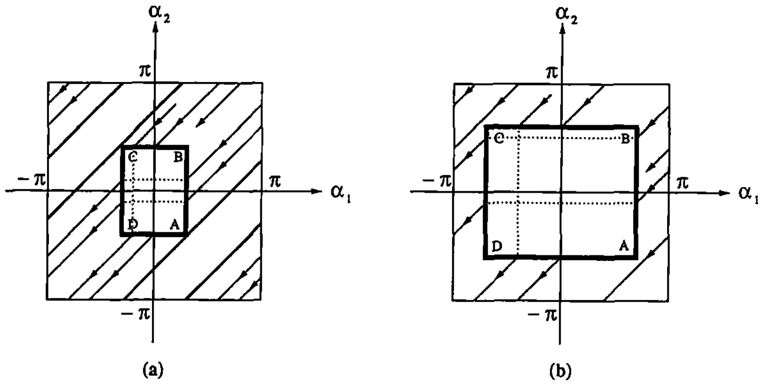

Fig. 9. The torus diffeomorphic to $M^0_{(E_1, E_2)}$ of the two noncoupled harmonic oscillators defined by (E_1, E_2) and parametrized by (α_1, α_2). The bold box represents the boundary of the harmonic wedge with $\beta = 3\pi/2$. The arrowed line of slope 1 is an orbit. The dotted lines represent the reflections. (a) $\varepsilon > 0$: the bold oblique lines delimit two invariant domains, one with only nonproper orbits and the other with only proper orbits (like the one represented). (b) $\varepsilon < 0$: there are no longer nonproper orbits.

Nevertheless, it is worthwhile inspecting in more detail the dynamics on the level set $M_{(E_1, E_2)}$ in the case $\varepsilon > 0$. Since $\bar{\alpha}_i < \pi$ for $i = 1$ and 2, there are orbits which never meet the rectangle (nonproper orbits corresponding to trajectories such as the one in Fig. 10). Then $M_{(E_1, E_2)}$ is made of two invariant manifolds with boundary (see Fig. 9a). Note that, as mentioned, there can also be nonproper orbits in the case $\beta = \pi/2$, but there are no level sets which contain these kinds of orbits together with proper orbits. It can be proved that the restriction of $M_{(E_1, E_2)}$ to nonproper orbits is diffeomorphic to a cylinder, and the restriction to proper orbits is diffeomorphic to a torus with two holes (that the orbits never encounter). Consequently each orbit of the system with $\beta = 3\pi/2$ and $\varepsilon > 0$ takes place on a piece of a torus. Hence the dynamics is that of an integrable system.

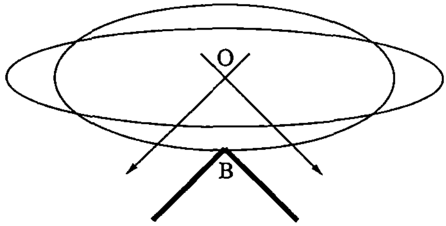

Fig. 10. Two nonproper trajectories of the harmonic wedge with $\beta = 3\pi/2$ and $\varepsilon > 0$. One of them is tangent to the corner at B. It belongs to the boundary of the invariant space of nonproper orbits.

The definition of pseudointegrability is then a question of point of view. If we consider only the topology of the level set, the system is not integrable, whereas if we take into account the dynamics on this level set, the system can be viewed as integrable.

3. THE DISCRETE-TIME APPROACH

3.1. Liouville's Theorem for Maps

Liouville's theorem has been extended to symplectic maps.[8] Since it seems very natural to study billiards systems through their bouncing map, we will now discuss integrability in the discrete-time approach.

Let T be a symplectic map which acts in a $2m$-dimensional phase space $M^{(T)}$. Let $F_1,..., F_m$ be m constants of motion (i.e., $F_i(T\mathbf{x}) = F_i(\mathbf{x})$, $\forall \mathbf{x} \in M^{(T)}$) in involution. Let $M_{\mathbf{k}}^{(T)}$ be a level set of these functions: $M_{\mathbf{k}}^{(T)} = \{\mathbf{x} \in M^{(T)} | F_i(\mathbf{x}) = k_i,\, i = 1,..., m\}$ and assume that the m 1-forms dF_i are independent at each point of $M_{\mathbf{k}}^{(T)}$. The theorem asserts:

(i) $M_{\mathbf{k}}^{(T)}$ is smooth and invariant under T.

(ii) If $M_{\mathbf{k}}^{(T)}$ is compact and connected, then it is diffeomorphic to the m-dimensional torus $\mathbb{T}^m = \{(\phi_1,..., \phi_m) \bmod 2\pi\}$.

(iii) There exist angle coordinate $\boldsymbol{\phi}$ and conjugate momenta $\mathbf{K}$ such that the map can be written in the form

$$T(\mathbf{K}, \boldsymbol{\phi}) = (\mathbf{K}, \boldsymbol{\phi} + \boldsymbol{\Omega}(\mathbf{K}))$$

The canonical change of variables from $(\mathbf{p}, \mathbf{q})$ to $(\mathbf{K}, \boldsymbol{\phi})$ can be done explicitly with the help of the same procedure as in Scheme 1. These results are valid for maps that are diffeomorphisms.

Consider now the particular case where the map T is a Poincaré section of a smooth Hamiltonian system of n degrees of freedom (so that $m = n - 1$). If the surface of section is smooth and transverse to the flow, i.e., the Poincaré map is defined correctly, then T is a symplectic diffeomorphism and the above theorem applies on it.

3.2. Bouncing Map of a Billiard

It is natural to study billiards through the bouncing map T, the return map with $M^{(T)} \equiv \partial M$ as the two-dimensional surface of section. There is one difficulty which just occurs in the study of billiards with nonsmooth boundary. If ∂Q is the union of a discrete number of smooth components connected at points where ∂Q is only C^k, then T is only C^{k-1}: if $k = 1$, T is a homeomorphism and if $k = 0$, T is discontinuous. Moreover, it can happen that the flow is not transverse to ∂M. At points of tangencies, the flow is not differentiable and T is discontinuous. Hence in general T is not a Poincaré section. It is given explicitly by several different maps T_{ij} (applied on points going from a smooth component i to another one j). Each one of these maps acts in a different domain of the phase space and the frontier between these domains is determined by the singularities of T. In what follows, we will say that the map T "consists of" the maps T_{ij}.

For such maps the question arises of the applicability of Liouville's theorem. The following three subsections will discuss the relationship between the concepts of integrability in the continuous-time approach and integrability of the discrete map.

3.3. AAV for the Bouncing Map of the Harmonic Wedge

3.3.1. Definitions. Since we will treat the problem in the harmonic wedge billiard, we will first discuss singularities in this particular case. There are three types of discontinuities of T, the existence and nature of which vary according to the parameters β and ε.

Type I. The angle β at B (see Fig. 11). As explained in ref. 9, if the angle β at a vertex B of a billiard is $\beta = \pi/n$, then T^n is continuous around B. The reason is that in this case the map is

$$\mathbf{x}_0 \in M \mapsto (\Pi \Phi_t)\, \mathbf{x}_0$$

where Π is the natural projection on Q defined by $\mathbf{x} = (\mathbf{q}, \mathbf{p}) \mapsto \mathbf{q}$, is continuous (the time needed for n reflections tends to zero when the orbit tends to B).

Type II. The trajectories tangent to the boundary (see Fig. 12). If $\varepsilon < 0$, there cannot be trajectories inside Q which are tangent to the boundary.

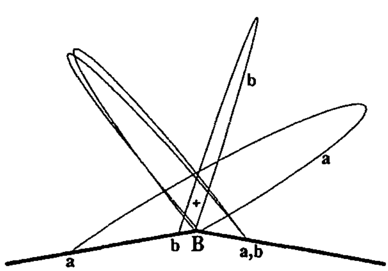

Fig. 11. Discontinuity due to the angle in the harmonic wedge with $\beta = 200°$ and $\varepsilon = 0.2$: two initially near trajectories (a and b) are separated. They start from the same point on the right side (denoted by a, b) with slightly different momenta such that a has its first collision close to B on the right side and b has its first collision close to B on the left side. Then, since reflections are different for both trajectories, they are separated and their next collisions occur at two distant points.

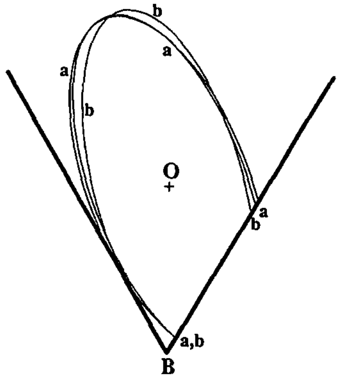

Fig. 12. Discontinuity due to tangencies in the harmonic wedge with $\beta = 60°$ and $\varepsilon = 1.2$: three initially near trajectories. Two of them, which remain close, are labeled by a and the other is labeled by b. They start from the same point on the right side (denoted by a, b) with slightly different momenta such that b has its first collision on the left side and the two other trajectories just "miss" the left side and have their first collision on the right side (one of them is tangent to the left side). After reflection, b has its second collision on the right side.

If $\beta > \pi$ and $\varepsilon > 0$, trajectories tangent to the boundary can be considered as nonproper. Then, discontinuities in the map T due to tangencies exist only if $\varepsilon > 0$ and $\beta < \pi$.

Type III. The trajectories tangent to the vertex B when $\beta > \pi$ (see Fig. 13). If $\varepsilon > 0$, trajectories tangent to the vertex can be considered as nonproper (Fig. 10). Then, discontinuities in the map T due to nonconvexity at the corner exist only if $\beta > \pi$ and $\varepsilon < 0$.

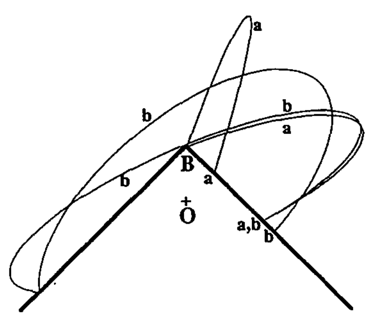

Fig. 13. Discontinuity due to tangencies at an angle greater than π in the harmonic wedge with $\beta = 3\pi/2$ and $\varepsilon = -0.2$: two initially near trajectories (a and b). They start from the same point on the right side (denoted by a, b) with slightly different momenta such that a has its first collision close to B on the right side and b, being tangent to the corner, has its first collision on the left side. Then they are separated and their collisions occur at distant points.

We also introduce Birkhoff-like variables (p, s) on ∂M. To do this, we define the oriented tangent to the boundary by $(\mu \cos(\beta/2), \sin(\beta/2))$ in the coordinates system of Fig. 2, with $\mu = -1$ on the left side and $\mu = 1$ on the right side. p is the projection of the momentum **p** on this tangent and s is the arc length parametrizing ∂Q (it is the Euclidean distance from the point on the boundary to B, with a negative sign on the left side). The map T expressed in these variables is symplectic. T consists of four maps: T_{ii}, $i = 1$ or 2, applying at points going from side i to itself directly (i.e., without encountering the other side before reaching side i) and T_{ij}, $i = 1$ or 2 and $j \neq i$, applying to points going from side j to side i directly. Note that we could also reduce T to two maps T_a and T_b using symmetry by identifying the sides: $T_a \Leftrightarrow T_{ii}$ with any $i \in \{1, 2\}$ and $T_b \Leftrightarrow T_{ij}$, with any $i, j \in \{1, 2\}, j \neq i$.

Here, the dimension of the phase space ∂M of the map T is 2, so that we have to find only one constant of motion.

3.3.2. Integrability and Discontinuous Maps: Case $\beta = \pi/2$. When the map T is the bouncing map of a billiard, it can happen that it is not a diffeomorphism. In this case, how is integrability of the billiard reflected in the map? In other words, can we still construct variables $(\mathbf{K}, \boldsymbol{\phi})$ having the right properties? We will answer this question for the integrable case $\beta = \pi/2$ of the harmonic wedge. Remember that T^2 is continuous around the singularity of type I at B. There are also discontinuities of type II, but they belong to the frontier between $M^{(a)}$ and $M^{(b)}$.

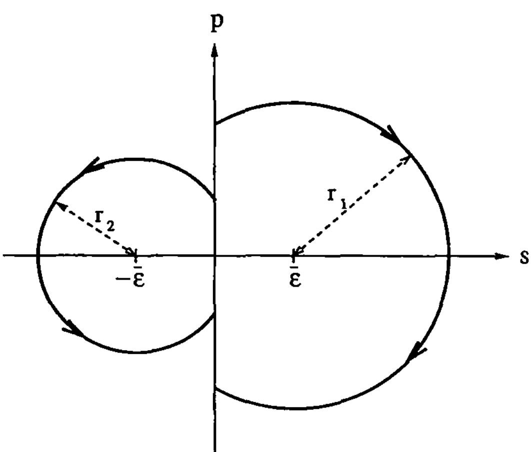

Fig. 14. The level set M_{E_1} of the map T when $\beta = \pi/2$, in Birkhoff's coordinates (p, s). Here $r_1 = (2E_1)^{1/2}$ and $r_2 = (2E - 2E_1)^{1/2}$. Orientation corresponds to that of the boundary.

We will first try to apply the construction of AAV for the map. The energies h_1^0 and h_2^0 of both modes are conserved (we use the same definitions as in Section 2.3.2). But the expressions of these functions in variables (p, s) are not the same in the whole phase space of T. On side 1,

$$2E_1 = p^2 + (s - \bar{\varepsilon})^2 \qquad \text{and} \qquad 2E_2 = 2E - p^2 - (s - \bar{\varepsilon})^2$$

and on side 2,

$$2E_1 = 2E - p^2 - (s + \bar{\varepsilon})^2 \qquad \text{and} \qquad 2E_2 = p^2 + (s + \bar{\varepsilon})^2$$

If we consider orbits in $M^{(a)}$ there are no difficulties. In $M^{(b)}$, we can consider, for instance, the energy of the first mode as the constant of motion. The resulting level set M_{E_1} is the union of two arcs of circles (see Fig. 14). As in the continuous-time approach, we have to paste the pieces of the level set to pursue the calculation. We will rather proceed in a clearer way by presenting a geometrical construction in Appendix A which gives finally the same definition of an angle ϕ for which T is expressed as a discrete rotation, and which can be reproduced for the pseudointegrable case when $\varepsilon > 0$.

Note that there is still another way to deal with the problem. Since the difficulty is to treat discontinuities due to the fact that we take into account the whole boundary of the billiard, a way to avoid problems is to consider only one of the sides to define the Poincaré section. We call T_i the map from side i to side i (with eventual collisions with the other side before going back to i). The energy h_i^0 is conserved by T_i, so that we may apply the construction of AAV. The level set $M_{h_i^0 = E_i}^{(T)}$ is the cycle Γ_i obtained for the continuous time (Fig. 4). Then we simply get

$$(K, \phi) = (I_i, \varphi_i)$$

This result is not surprising since the sides of the wedge are straight lines $\varphi_1 = 2\pi$ and $\varphi_2 = 2\pi$ in the angle coordinates constructed for the continuous time.

3.3.3. Integrability and Poincaré Sections: Case $\epsilon = 0$. In the particular case where T is a Poincaré section of Hamiltonian system, what is the relationship between the AAV $(\mathbf{K}, \boldsymbol{\phi})$ constructed for T in the $(2n-2)$-dimensional surface of section and the one $(\mathbf{I}, \boldsymbol{\varphi})$ constructed for the Hamiltonian flow in continuous time in the $2n$-dimensional phase space? In general, one cannot expect that the surface of section defining T is given by $\varphi_i = \text{const}$, $i = 1, \ldots, n-1$, in which case we would have $\phi_i = \varphi_i$,

$i = 1, \ldots, n-1$ (as in case $\beta = \pi/2$ above). $\boldsymbol{\phi}$ can be a more complicated function of $\boldsymbol{\varphi}$.

We will answer this question for the integrable case $\varepsilon = 0$ of the harmonic wedge by applying Scheme 1 for maps and comparing the result with the one obtained for the continuous time.

Since in this case all trajectories go from one side to the other one, the map T consists of T_b only and is a diffeomorphism. In coordinates $(p_{\|}, q_{\|} \equiv (r, p_r) \equiv (\mu p, \mu s)$ (which identifies the two sides) the constant of motion $|p_\theta|$ is given by $|p_\theta| = q_{\|}(2E - p_{\|}^2 - q_{\|}^2)^{1/2}$. The level set $M_{|p_\theta| = |L|}$ is the cycle Γ_r introduced above for the construction of AAV for the continuous time. As a consequence we have

$$K = I_2 = \frac{E - |L|}{2}$$

Note that the energy E is a parameter of the map and no longer a constant of motion. Hence the generating function can be expressed with the one obtained in the case of the continuous time: $S(K, q_{\|}) = S_r(J_1(E, K),$

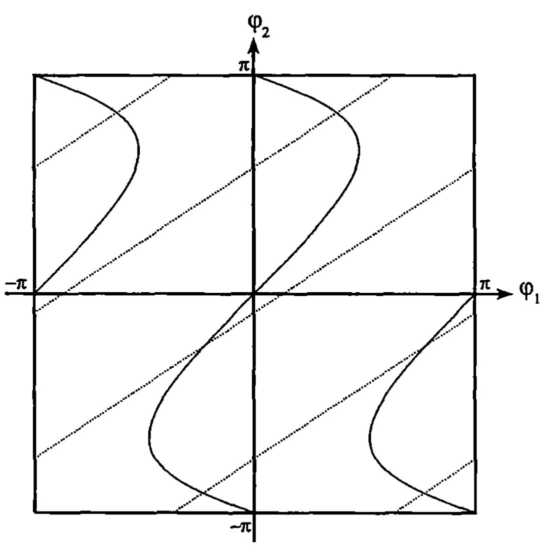

Fig. 15. The torus $M^c_{(E, |L|)}$ defined by (I_1, I_2) and parametrized by (φ_1, φ_2) which is isomorphic to the level set $M_{(E, |L|)}$ of the harmonic wedge when $\varepsilon = 0$. The two sides of the wedge are represented by the two curves $\varphi_1(\varphi_2)$. These curves are parametrized by I_1 and I_2 which depend, at a fixed energy E, on $|L|$. Here $\beta = 60°$ and $|L| = 0.67$. One orbit is represented on the figure [oblique dotted lines of slope $\tan(\Omega)$]. Since $\tan(\Omega) = 2\beta/\pi = \frac{1}{3}$ is rational, all orbits are periodic.

$J_2(E, K), q_{||})$ where $J_1(E, K) = |L| = E - 2K$ and $J_2(E, K) = K$. We obtain the angle

$$\phi = \frac{\partial S}{\partial K} = \frac{\partial J_1}{\partial K}\frac{\partial S_r}{\partial J_1} + \frac{\partial J_2}{\partial K}\frac{\partial S_r}{\partial J_2}$$

which leads to

$$\phi = -2\frac{\beta}{\pi}\varphi_1 + \varphi_2$$

We can now interpret this change of variables with the help of what has been done for continuous time. Let us examine the shape of the boundary of the billiard in variables (φ_1, φ_2) at fixed values of (I_1, I_2). Since each side can be written as a function of φ_2, as seen in Fig. 15, the boundary in variables (φ_1, φ_2) can be parametrized by the coordinate $\phi = -2(\beta/\pi)\varphi_1 + \varphi_2$. This corresponds to a projection of the surface of section on the surfaces $\varphi_1 = 0$ and $\varphi_1 = \pi$ in the direction parallel to the flow. In other words, this corresponds to a change of surface of section defining a new Poincaré section since the resulting map in coordinates (K, ϕ) may also be interpreted as a return map.

3.4. Pseudointegrability Reflected in the Map: Case $\beta = 3\pi/2$

We recall that pseudointegrability has been defined through the structure of the phase space, therefore it is a property visible in the continuous-time approach. How will pseudointegrability manifest itself in the discrete-time system?

We will answer this question for the pseudointegrable case $\beta = 3\pi/2$ of the harmonic wedge. We use the same definitions as in Section 2.4.

3.4.1. Case $\epsilon > 0$. First we consider the situation where $\varepsilon > 0$. As we have seen, the discontinuities of type III are not manifested in the map. These discontinuities are represented on Fig. 16 by points A and C: these points belong to the invariant subspace of nonproper orbits. The only discontinuity is of type I, which is represented by B. Let us examine in more detail how orbits arriving close to the vertex are separated.

It is easy to follow an orbit on the torus $M^0_{(E_1, E_2)}$ using Fig. 16a. Two parallel orbits arriving close to B, the first on side 1 (segment BC) and the second on side 2 (segment AB), are separated since the first is sent by the reflection close to A on segment DA and the second close to C on segment CD. Then the geodesic flow sends the first close to A on segment AB and

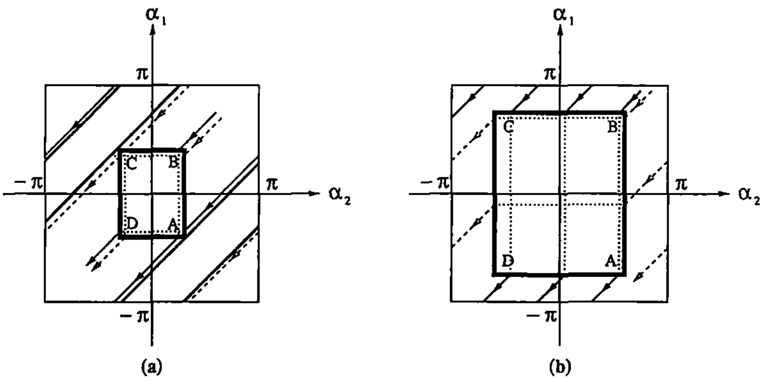

Fig. 16. Same representation as in Fig. 9. (a) The process responsible for the continuity of T^2 when $\beta = 3\pi/2$ and $\varepsilon > 0$: two near orbits arriving close to B are again near after two collisions. (b) $\varepsilon < 0$: this is not longer the case, because the orbits have other collisions before they can be repasted.

the second close to C on segment BC. At the end, the reflection sends the first close to D on segment CD and the second close to D on segment DA. Hence two near orbits arriving close to B are separated by the first reflection but are pasted together (i.e., they are again near) after a finite time. From this, we can deduce that T^2 is continuous around B.

Here we have a new counterexample of the heuristic rule stated in ref. 9. We said in that paper that there could exist billiards where a certain power of T is continuous due to nonlocal properties of the flow: two orbits first separated by the angle can be pasted later (after a nonvanishing time) if a second passage through a discontinuity systematically occurs which cancels the effect of the first. We presented an example where this cancellation was due to the shape of the boundary. Here we have a second example, where this cancellation is due to the potential.

Now, we come back to the harmonic wedge. Thanks to the fact that the proper orbits stay in a restriction of $M_{(E_1, E_2)}$, we can make the same reasoning as for the case $\beta = \pi/2$ (see Appendix B). Hence we can construct AAV for the bouncing map. But now the map $\mathbf{x}_0 \in M \mapsto (\Pi\Phi_t)\, \mathbf{x}_0$ is discontinuous anyway. That is why the system is defined as being pseudo-integrable in the continuous-time approach, but is clearly integrable in the discrete-time approach, which automatically takes into account the dynamics on the level set. In the discrete-time approach, the integrable nature of the system is more evident.

3.4.2. Case $\varepsilon < 0$. In the situation where $\varepsilon < 0$, the discontinuities of type III play an important rôle. Indeed, if we try to follow orbits arriving close to the vertex B in Fig. 16b, we see that these orbits encounter the boundary before coming back to B. so that they will not be pasted later: the discontinuities of type III forbid the process of cancellation of the first discontinuity of type I.

By the same type of construction as done in Appendixes A and B for the previous cases, we can see in Appendix C that the dynamics is that of an interval exchange transformation with four intervals, whereas it was two intervals in the preceding results. We could say that T is pseudointegrable, being an interval exchange transformation of more than two intervals (as in billiards in rational polygons; see ref. 7).

4. CONCLUSION

In the continuous-time approach, we have seen that Liouville's theorem is still valid for billiards, with slight changes in some statements. It is not always true that the level sets are isomorphic to tori: the presence of singularities of nonzero index in the vector fields tangent to the level sets makes the theorem not valid. In the integrable cases of billiards, the changes of variables leading to AAV are, in general, nondifferentiable isomorphisms: the level sets are only isomorphic to the torus $\mathbb{T}^2$.

If the construction of AAV for the continuous time may be performed (i.e., if we have checked that the level set is isomorphic to a torus), we get a Hamiltonian formulation of the billiard problem without having to deal explicitly with the reflections: the change of variables leading to AAV allows us to obtain the flow Φ_t of the billiard in variables $(\mathbf{I}, \boldsymbol{\varphi})$ as a proper function of time. Then the question arises of whether this single structure is preserved when we perturb the integrable case. The answer differs for the two cases of the harmonic wedge that we treated.

In the discrete-time approach, we have seen that if T is a diffeomorphism, the change of variables leading to the AAV of the map may be interpreted as a change of the Poincaré surface of section. It also appeared that considering the whole boundary as the surface of section is not the better choice to study the system. We also underlined a deep difference between continuous time and discrete time in the pseudointegrable case, since the map corresponding to a pseudointegrable case (in the sense of the continuous time) may take the form of an integrable map. This leads to differences of the definition of pseudointegrability for the two points of view.

APPENDIX A. INTEGRABILITY OF THE MAP IN THE CASE $\beta = \pi/2$

On the torus $M^0_{(E_1, E_2)}$, in coordinates (α_1, α_2), we define the points $A \equiv (-\bar{\alpha}_1, \bar{\alpha}_2)$, $E \equiv (-\bar{\alpha}_1, 2\bar{\alpha}_1 - \bar{\alpha}_2)$, $B \equiv (-\bar{\alpha}_1, -\bar{\alpha}_2)$, and $C \equiv (\bar{\alpha}_1 - \bar{\alpha}_2)$,

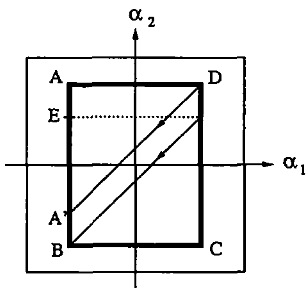

Fig. 17. Using the same representation as in Fig. 5a, we show how to cut off the line ABC in order to interpret T as an interval exchange transformation when $\beta = \pi/2$. Here E is the preimage of B and A′ is the image of A (A is a point on side 2). T sends AE to A′B, EB to BC (then it sends AB to A′C), and BC to AA′. Then T exchanges two intervals: it is a discrete rotation.

where $\bar{\alpha}_i = \alpha_0(E_i)$ (see Fig. 17). We suppose that $\bar{\alpha}_1 < \bar{\alpha}_2$ (otherwise permute the indexes denoting the sides). AB represents side 2, BC represents side 1, and E is the preimage of B. Since the slope of the orbits in this representation is 1, length(BC) = length(EB); this is a crucial fact in this construction. We can parametrize univoquely the boundary with the help of the variable $\phi \in [0, 2\pi[$ defined as follows:

$$\phi = \begin{cases} \dfrac{\pi}{\bar{\alpha}_1 + \bar{\alpha}_2} (\bar{\alpha}_2 - \alpha_2) & \text{if} \quad \alpha_1 = -\bar{\alpha}_1 \\ \dfrac{\pi}{\bar{\alpha}_1 + \bar{\alpha}_2} (2\bar{\alpha}_2 + \bar{\alpha}_1 + \alpha_1) & \text{if} \quad \alpha_2 = -\bar{\alpha}_2 \end{cases}$$

The action of T on ϕ is given by

$$\phi \mapsto \phi + 2\pi \frac{\bar{\alpha}_1}{\bar{\alpha}_1 + \bar{\alpha}_2}$$

Then the map T at fixed value of (E_1, E_2) can be written as a discrete rotation and has the form of an integrable map.

APPENDIX B. INTEGRABILITY OF THE MAP IN THE CASE $\beta = 3\pi/2$, $\epsilon > 0$

On the torus $M^0_{(E_1, E_2)}$, in coordinates (α_1, α_2), we define the points $A \equiv (\bar{\alpha}_1, -\bar{\alpha}_2)$, $E \equiv (\bar{\alpha}_1, \bar{\alpha}_2 - 2\bar{\alpha}_1)$, $B \equiv (\bar{\alpha}_1, \bar{\alpha}_2)$, and $C \equiv (-\bar{\alpha}_1, \bar{\alpha}_2)$, where $\bar{\alpha}_i = \arccos(\bar{\varepsilon}/\sqrt{2E_i})$ (see Fig. 18a). We suppose that $\bar{\alpha}_1 < \bar{\alpha}_2$ (otherwise permute the numbers of the sides). With these definitions, the dynamics and

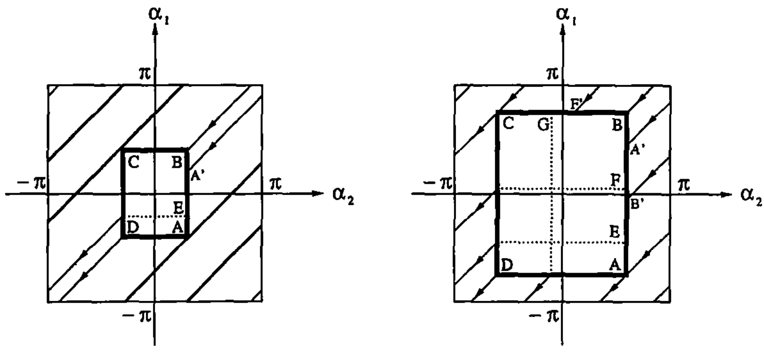

Fig. 18. Using the same representation as in Fig. 9, we show how to cut off the line ABC in order to interpret T as an interval exchange transformation when $\beta = 3\pi/2$. (a) $\varepsilon > 0$. Here E is the preimage of B and A'is the image of A. The map T sends AB to A'C and BC to AA'. Then T exchange two intervals. (b) $\varepsilon < 0$. Here E is the preimage of B, B' is the image of B (interpreted as a point on side 2), F is the preimage of A, A' is the image of A, F' is the image of B (interpreted as a point on side 1), and G is the preimage of C. The map T sends AF to A'F', FB to AB', BG to F'C, and GC to B'A'. Then T exchanges four intervals.

the significance of the segments AE, EB, and BC are the same as in Appendix A when $\beta = \pi/2$. Hence we find that we can parametrize univocally the boundary with the help of the variable $\phi \in [0, 2\pi[$ defined as follows:

$$\phi = \begin{cases} \dfrac{\pi}{\bar{\alpha}_1 + \bar{\alpha}_2} (\bar{\alpha}_2 + \alpha_2) & \text{if} \quad \alpha_1 = -\bar{\alpha}_1 \\ \dfrac{\pi}{\bar{\alpha}_1 + \bar{\alpha}_2} (2\bar{\alpha}_2 + \bar{\alpha}_1 - \alpha_1) & \text{if} \quad \alpha_2 = -\bar{\alpha}_2 \end{cases}$$

We get the same conclusion for the action of the map T on the variable ϕ.

We took advantage of the fact that the billiard with $\beta = 3\pi/2$ and $\bar{\varepsilon} = d > 0$ can be interpreted as the "exterior problem" of the billiard with $\beta = \pi/2$ and $\bar{\varepsilon} = -d < 0$. It is easy to prove, using coordinates (α_1, α_2), that to each orbit of the first billiard there corresponds univocally an orbit of the second. Calling T_{ext} the bouncing map of the first billiard and T_{int} that of the second, we have

$$T_{\text{ext}} \mathbf{x}_0 = I T_{\text{int}} I \mathbf{x}_0$$

where I is the map corresponding to the inversion of the momentum: $\mathbf{p} \mapsto -\mathbf{p}$. The map I sends the points A, B, C, D introduced for $\beta = \pi/2$ to, respectively, the points A, B, C, D introduced for $\beta = 3\pi/2$.

APPENDIX C. INTERVAL EXCHANGE TRANSFORMATION IN THE CASE $\beta = 3\pi/2$, $\epsilon < 0$

For a fixed value of (E_1, E_2), in coordinates (α_1, α_2), we define the points $\mathrm{A} \equiv (\bar{\alpha}_1 - \bar{\alpha}_2)$, $\mathrm{E} \equiv (\bar{\alpha}_1, \bar{\alpha}_2 - 2\bar{\alpha}_1)$, $\mathrm{F} \equiv (\bar{\alpha}_1, 2\pi - 2\bar{\alpha}_1 - \bar{\alpha}_2)$, $\mathrm{B} \equiv (\bar{\alpha}_1, \bar{\alpha}_2)$, $\mathrm{G} \equiv (2\pi - 2\bar{\alpha}_2 - \bar{\alpha}_1, \bar{\alpha}_2)$, and $\mathrm{C} \equiv (-\bar{\alpha}_1, \bar{\alpha}_2)$, where $\bar{\alpha}_i = \arccos(\bar{\varepsilon}/\sqrt{2E_i})$ (see Fig. 18b). We suppose that $\bar{\alpha}_1 < \bar{\alpha}_2$ (otherwise permute the numbers of the sides). E is the preimage of B, F is the preimage of A, and G is the preimage of C. We can introduce a variable ϕ which parametrizes the boundary as before. This time, T exchanges the four intervals AF, FB, BG, and GC. It is not a rotation (interval exchange transformation of only two intervals). The discontinuities play a major role in the cutting of the interval. When $\varepsilon < 0$ we must take into account the preimages of points A and C. Note that, due to these discontinuities, the billiard with $\beta = 3\pi/2$ and $\bar{\varepsilon} = -d < 0$ is not the exterior problem of the billiard with $\beta = \pi/2$ and $\bar{\varepsilon} = d > 0$.

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